Journal of Statistical Physics, Vol. 77, Nos. 3/4, 1994

The Dimensions of Some Self-Affine Limit Sets in the Plane and Hyperbolic Sets

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Received March 7, 1994; final April 25, 1994

In this article we compute the Hausdorff dimension and box dimension (or capacity) of a dynamically constructed model similarity process in the plane with two distinct contraction coefficients. These examples are natural generalizations to the plane of the simple Markov map constructions for Cantor sets on the line. Some related problems have been studied by different authors; however, those results are directed toward generic results in quite general situations. This paper concentrates on computing explicit formulas in as many specific cases as possible. The techniques of previous authors and ours are correspondingly very different. In our calculations, delicate number-theoretic properties of the contraction coefficients arise. Finally, we utilize the results for the model problem to compute the dimensions of some affine horseshoes in $\mathbb{R}^n$, and we observe that the dimensions do not always coincide and their coincidence depends on delicate number-theoretic properties of the Lyapunov exponents.

KEY WORDS: Hausdorff dimension; box dimension; self-affine; PV numbers; random geometric series; horseshoes; infinitely convolved Bernoulli measure.

1. INTRODUCTION

The notions of Hausdorff dimension and box dimension play an important role in describing the concept of "size" of sets in the plane. They are particularly useful in analyzing sets of Lebesgue measure zero.

A situation that is relatively well understood is that of Cantor sets in the line that are constructed dynamically from simple Markov ("cookie

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0022-4715/94/1100-0841$07.00/0

cutter") transformations. Let $I_0, I_1 \subset [0, 1]$ be two disjoint closed subintervals. A simple Markov map is a $C^{1+\alpha}$ map $g: I_0 \cup I_1 \to \mathbb{R}$ such that (i) $|g'| > 1$ and (ii) $g(I_0) = g(I_1) = [0, 1]$. Then $\Lambda = \{x \in [0, 1]: g^k(x) \in [0, 1]$ for $k = 0, 1, 2,...\}$ is the associated Cantor set. In this particular case, considerable advantage is gained from the one-dimensionality, and more specifically from the conformal nature of the transformation. The situation becomes more complicated when nonconformal and higher-dimensional analogs are considered.

In this paper we first deal with the simplest "model problem" in which we are concerned with a Cantor set in the plane constructed from two similar affine contractions. Our set is described by a finite number of parameters (the horizontal contraction λ_2, the vertical contraction λ_1, the position of the images, etc.), and it becomes a very natural question to ask about the dependence of the Hausdorff dimension and box dimension on these parameters.

For this important special problem, we compute the box dimension in all cases and the Hausdorff dimension in a large number of cases. Our pivotal observation is that if $\alpha = \max\{\lambda_1, \lambda_2\} < 1/2$, then the Hausdorff dimension and box dimension are equal and depend only on α.

In subsequent sections, we present necessary conditions on $\alpha > 1/2$ such that the Hausdorff dimension and box dimension will again be equal. This problem seems to be intimately connected with very deep results in number theory and harmonic analysis on PV numbers due to Garcia and Erdos. We introduce a new class of numbers (**GE** numbers) that are a subset of, but more suitable for our purposes than, the well-known class of numbers studied by Kahane and Salem.[11] Previous applications of these ideas to Hausdorff dimension estimates seem to have been limited to graphs of Weierstrass-like functions.[12, 16] Along the way, we construct the first one-parameter family of Cantor sets whose Hausdorff dimension and box dimension do not coincide.

We then describe how our ideas may be applied to various generalizations of this problem.

Finally, we utilize the results for the model problem to compute the dimensions of some linear horseshoes in $\mathbb{R}^n$. We construct an easy example to show that the Hausdorff dimension of some linear horseshoes depends on the *geometry* of the horseshoe, and hence that the Hausdorff dimension of a hyperbolic set cannot be computed from just dynamical quantities, i.e., entropies and Lyapunov exponents. Moreover, we construct examples of linear horseshoes *all having the same geometry*. In some of the examples the dimensions coincide, and in other examples the dimensions do not coincide. The largest Lyapunov exponent of the latter examples is a reciprical of a PV (badly approximable) number. *This indicates that the Hausdorff* *dimensions of hyperbolic sets can depend on delicate number-theoretic properties of the Lyapunov exponents.* This is in contrast to the Hausdorff dimension of an invariant measure for a surface diffeomorphism, where the dimension can be computed solely in terms of the Lyapunov exponents and the measure-theoretic entropy.[21] We conclude that the Hausdorff dimension of an invariant measure supported on a hyperbolic set is a much more natural and tractible quantity to study than the Hausdorff dimension of a hyperbolic set.

2. DESCRIPTION OF MODEL PROBLEM

Let $R_0, R_1 \subset I$ be two disjoint boxes in the unit square $I \subset \mathbb{R}^2$ (aligned with the axes of I) each having the same height, $0 < \lambda_1 < 1$, and the same width, $0 < \lambda_2 < 1$. See Fig. 1. For convenience assume $0 < \lambda_1 \leqslant \lambda_2 < 1$. If this is not the case, we need only interchange the two coordinates of the original square.

Consider the two affine maps $A_0 \colon I \to R_0$ and $A_1 \colon I \to R_1$ that contract the unit square by λ_1 in the vertical direction and λ_2 in the horizontal direction and place the resulting images in R_0 and R_1, respectively. Denote by F the self-affine limit set induced by A_0 and A_1, i.e., the unique compact set invariant under A_0 and A_1.[5] The step-n approximation of this limit set consists of the 2^n rectangles obtained by applying all n-fold compositions of A_0 and A_1 to I (see Fig. 1).

Notation. Let $\dim_B F$ and $\dim_H F$ denote the box dimension and Hausdorff dimension, respectively, of the set F. Let $\pi_k F \subset [0, 1]$ denote the projection of the limit set F onto the λ_k axis, for $k = 1, 2$. Full definitions and properties are given in Appendix A.

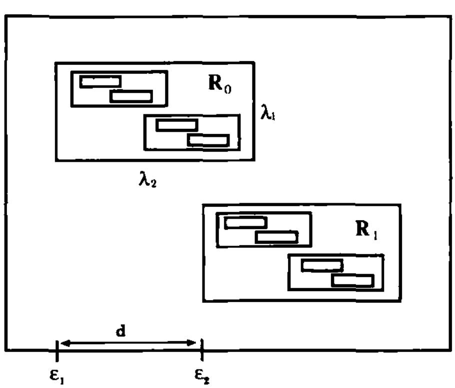

Fig. 1.

We first present a table of calculated values of the dimensions of F that follow from the results in this paper. In cases 2–5 we assume that $\lambda_1 < \lambda_2$.

Explicit Formulas. With the hypotheses above, we have the following situations:

Case 0:

$$\lambda = \lambda_1 = \lambda_2 \Rightarrow \dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F = \frac{\log 2}{\log(1/\lambda)}$$

Case 1:

$$\pi_2 R_0 = \pi_2 R_1 \Rightarrow \dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F = \frac{\log 2}{\log(1/\lambda_1)}$$

(R_0 is directly on top of R_1—degenerate case.)

Case 2:

$$\lambda_2 \leqslant \frac{1}{2} \Rightarrow \dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F = \frac{\log 2}{\log(1/\lambda_2)}$$

Case 3:

$$\lambda_2 = 2^{-1/p}, \quad p \in \mathbb{N} \Rightarrow \dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F = \frac{\log(2\lambda_2/\lambda_1)}{\log(1/\lambda_1)}$$

Case 4: There exists $0 < \gamma < 1$ such that for almost all λ_2 with

$$\gamma < \lambda_2 < 1 \Rightarrow \dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F = \frac{\log(2\lambda_2/\lambda_1)}{\log(1/\lambda_1)}$$

Case 5:

$$\lambda_2 \geqslant \frac{1}{2} \Rightarrow \dim_{\mathrm{B}} F = \frac{\log(2\lambda_2/\lambda_1)}{\log(1/\lambda_1)}$$

We prove the equality of the Hausdorff dimension and box dimension in the above cases by finding a number-theoretic sufficient condition on λ_2 that ensures the dimensions coincide. It is here that we encounter a fundamental dichotomy between the cases $0 < \lambda_2 < \frac{1}{2}$ and $\frac{1}{2} < \lambda_2 < 1$. The condition is always satisfied for $0 < \lambda_2 < \frac{1}{2}$. The case when $\frac{1}{2} < \lambda_2 < 1$ is much more subtle. The sufficient condition is violated when λ_2 is the reciprocal of a Pisot–Vijayarghavan (PV) number. Recall that PV numbers are algebraic integers whose conjugates all lie within the unit circle.[19,9,4]

It is not clear, *a priori*, whether the number-theoretic property of λ_2 that we require to show coincidence of the dimensions is really necessary or just a manifestation of our proof. In ref. 16 the authors show that for these problematic PV numbers, the equality between the two types of dimension actually breaks down:

Proposition.[16] For $\lambda_1 = \frac{1}{2}$ and λ_2 equal to the reciprocal of a PV number, there exists certain configurations such that $\dim_H F < \dim_B F$.

In ref. 16 the authors compute the dimension of graphs of Weierstrass-like functions. It just so happens that in this very special case the sets F reduce to graphs of Weierstrass-like functions (modulo countable sets) and hence the results in ref. 16 are applicable to our case. One can combine the result in ref. 16 with Proposition 2 to construct the first one-parameter family of Cantor sets whose Hausdorff dimension and box dimension do not coincide.

We conjecture that for (Lebesgue) almost all $\frac{1}{2} < \lambda_2 < 1$, the Hausdorff dimension coincides with the box dimension and is given (for non-degenerate configurations) by the formula in case 4 in the list of explicit formulas. This conjecture would imply that for almost all $0 < \lambda_2 < 1$, the Hausdorff dimension of the limit set of the model similarity process coincides with the box dimension. Morover, we conjecture that for a generic similarity process (i.e., the boxes need not be aligned with the axes of I), the Hausdorff dimension coincides with the box dimension.

We wish to quickly dispose of the degenerate configurations where $\pi_2 R_0 = \pi_2 R_1$, i.e., R_0 is directly on top of R_1. In these cases, the limit sets are easily seen to be uniform Cantor subsets of a vertical line having exponent λ_1, and hence $\dim_H F = \dim_B F = \log 2/\log(1/\lambda_1)$. We will henceforth assume that all configurations are nondegenerate.

3. CALCULATION OF $\dim_B F$

In this section, we shall present a direct computation of the box dimension of F in all cases. We remark that Falconer[8] has related results. We begin with some notation.

Notation. Denote the rectangles under n iterations of the affine maps by $R_{(i_1,\dots,i_n)} = A_{i_n} \cdots A_{i_1} I \subset I$, $(i_1,\dots, i_n) \in \{0, 1\}^n$.

Let $\pi_k F \subset [0, 1]$ denote the projection of the limit set F onto the λ_k axis, for $k = 1, 2$. We begin with a simple characterization of this projection that will prove useful later.

Proposition 1. We have the following explicit representation for the projection of the limit set F onto the λ_2 axis:

$$\pi_2 F = \left\{ \sum_{k=0}^{\infty} [\varepsilon_1 + i_{k+1}(\varepsilon_2 - \varepsilon_1)] \lambda_2^k; (i_1, i_2, \ldots) \in \{0, 1\}^{\mathbb{N}} \right\}$$

$$= \left\{ \frac{\varepsilon_1}{1-\lambda_2} + \underbrace{(\varepsilon_2 - \varepsilon_1)}_{d} \sum_{k=0}^{\infty} i_{k+1} \lambda_2^k; (i_1, i_2, \ldots) \in \{0, 1\}^{\mathbb{N}} \right\}$$

$$\subset \left[\frac{\varepsilon_1}{1-\lambda_2}, \frac{\varepsilon_2}{1-\lambda_2} \right] \equiv J$$

Proof. The affine maps A_0, A_1 on the square project to affine maps on the λ_2 axis B_0, B_1: $[0, 1] \to [0, 1]$ of the form

$$B_0(x) = \varepsilon_1 + \lambda_2 x \qquad \text{and} \qquad B_1(x) = \varepsilon_2 + \lambda_2 x$$

where $0 < \varepsilon_1, \varepsilon_2 < 1$ are the left endpoints of the intervals $\pi_2 \mathbf{R}_0$, $\pi_2 \mathbf{R}_1$, respectively. By induction, the left endpoint of $\pi_2 R_{(i_1, \ldots, i_n)}$ is given by $B_{i_n} \cdots B_{i_1} 0 = \sum_{k=0}^{n-1} [\varepsilon_1 + i_{k+1}(\varepsilon_2 - \varepsilon_1)] \lambda_2^k$. The lemma follows by taking limits. ∎

The following consequence will be useful in our estimates.

Corollary 1.1. 1. If $\frac{1}{2} \leqslant \lambda_2 < 1$, then $\pi_2 F = J$.

2. If $0 < \lambda_2 < \frac{1}{2}$, then $\pi_2 F =$ Cantor set.

Remarks. 1. If $\lambda_2 = \frac{1}{2}$, then Corollary 1.1 corresponds to computing the dyadic expansion of numbers in J.

2. When $\lambda_2 = \frac{1}{3}$, then Corollary 1.1 corresponds to the construction of a copy of the middle third Cantor set.

Proof of Corollary 1.1. If $\frac{1}{2} \leqslant \lambda_2 < 1$, then the expression in Proposition 1 is the β expansion for $\beta = 1/\lambda_2$. It is easy to see that every $x \in J$ has a β expansion, hence $\pi_2 F = J$. If we take the orbit of x under the associated expanding map (making arbitrary choices of the expanding map when the domains overlap), then the itinerary of the orbit gives the corresponding β expansion.

For the case $\lambda_2 < \frac{1}{2}$, we need only observe that the set is affinely equivalent to the standard Cantor set

$$\sum_{k=0}^{\infty} i_{k+1} \lambda_2^k; \qquad (i_1, i_2, \ldots) \in \{0, 1\}^{\mathbb{N}} \tag{3.1}$$

after scaling (by $d > 0$) and translating [by $\varepsilon_1/(1-\lambda_2)$]. ∎

Notation. Let F_n denote the union of the disjoint rectangles $R_{(i_1,\dots,i_n)}$ for $(i_1,\dots,i_n)\in\{0,1\}^n$.

Proposition 2. If $\frac{1}{2}\leqslant\lambda_2<1$, then $\dim_B F=\log(2\lambda_2/\lambda_1)/\log(1/\lambda_1)$.

Proof. Consider the obvious cover of the 2^n rectangles in F_n by squares with sides of length λ_1^n. Each of these rectangles $R_{(i_1,\dots,i_n)}$ has width λ_2^n and height λ_1^n and is covered by squares with side λ_1^n aligned in a row. The minimum number of squares needed for each such rectangle is $[\lambda_2^n/\lambda_1^n]$. We see from Corollary 1.1 that for all rectangles $R_{(i_1,\dots,i_n)}$, the projection $\pi_2(F\cap R_{(i_1,\dots,i_n)})$ contains an interval of length $|J|$ length $(\pi_2(R_{(i_1,\dots,i_n)}))=|J|\,\lambda_2^n=d\lambda_2^n/(1-\lambda_2)$. Hence, of those squares in the cover of each $R_{(i_1,\dots,i_n)}$, the proportion of the squares in the cover of $R_{(i_1,\dots,i_n)}$ required to cover $R_{(i_1,\dots,i_n)}\cap F$ is at least the ratio of the length of the contained interval of the rectangle, i.e.,

$$\frac{d\lambda_2^n/(1-\lambda_2)}{\lambda_2^n}=\frac{d}{1-\lambda_2} \tag{3.2}$$

Let $N(\delta)$ denote the minimum number of δ-squares needed to cover F. Our above reasoning tells us that when $\delta=\lambda_1^n$, there are 2^n rectangles in F_n, each of which is covered by λ_2^n/λ_1^n squares of size λ_1^n. To cover F, the proportion given by (3.2) is needed, i.e.,

$$N(\lambda_1^n)=2^n\frac{\lambda_2^n}{\lambda_1^n}\frac{1}{1-\lambda_2}$$

The proposition follows from Lemma A1 in Appendix A. ∎

Proposition 3. If $0<\lambda_2<\frac{1}{2}$, then $\dim_B F=\log 2/\log(1/\lambda_2)$.

Proof. The set $\pi_2 F$ was seen to be a Cantor set in Corollary 1.1. However, it is possible that the projections onto the λ_2 axis of the 2^n rectangles in F_n may not be disjoint (this anomaly occurs because the Cantor set F is really defined through the affine maps A_0, A_1 and not the covers F_n). However, our *formula* in Proposition 1 for $\pi_2 F$ gives a construction that uses 2^n *disjoint* intervals of length $d\lambda_2^n/(1-\lambda_2)$ at step n. Assume that the original boxes are given by $R_0=[\varepsilon_1,\varepsilon_1+\lambda_2]\times J_1$ and $R_1=[\varepsilon_2,\varepsilon_2+\lambda_2]\times J_2$, where J_1, J_2 are intervals in the vertical axis. Once we have defined the maps $A_0\colon I\to R_0$ and $A_1\colon I\to R_1$, we replace these boxes by the new boxes

$$R_0^*=\left[\frac{\varepsilon_1}{1-\lambda_2},\frac{\varepsilon_1}{1-\lambda_2}+\frac{d\lambda_2}{1-\lambda_2}\right]\times J_1 \qquad\text{and}\qquad R_1^*=\left[\frac{\varepsilon_2-d\lambda_2}{1-\lambda_2},\frac{\varepsilon_2}{1-\lambda_2}\right]\times J_2$$

These project (on the horizontal axis) onto the two *disjoint* intervals

$$\left[\frac{\varepsilon_1}{1-\lambda_2}, \frac{\varepsilon_1}{1-\lambda_2} + \frac{d\lambda_2}{1-\lambda_2}\right] \quad \text{and} \quad \left[\frac{\varepsilon_2 - d\lambda_2}{1-\lambda_2}, \frac{\varepsilon_2}{1-\lambda_2}\right]$$

whose endpoints correspond to the sequences

$$\begin{aligned}(i_1, i_2,\ldots) &= (0, 0, 0, 0,\ldots)\\ &= (0, 1, 1, 1,\ldots)\\ &= (1, 0, 0, 0,\ldots)\\ &= (1, 1, 1, 1,\ldots)\end{aligned}$$

These intervals give the "standard" construction of the Cantor set in the projection $\pi_2 F$, as can be seen by Proposition 1.

We can define new rectangles $R^*_{(i_1,\ldots,i_n)} = A_{i_n} \cdots A_{i_2} R^*_{i_1}$. The 2^n rectangles at step n are all disjoint since $R^*_l \subset R_l$ for $l = 1, 2$, and thus $R^*_{(i_1,\ldots,i_n)} \subset R_{(i_1,\ldots,i_n)}$. We let F^*_n denote the union of these 2^n rectangles. Clearly the limit set F is contained in the union of these rectangles for all n. The projection onto the horizontal axis consists of 2^n intervals of length $c\lambda_2^n$, where $c = d/\lambda_2^n$.

Let

$$m = \left[\frac{\log[(1/c)\,\lambda_1^n]}{\log \lambda_2}\right]$$

Our goal is to find "optimal covers" of $F \cap F^*_m$ by squares of length λ_1^n. It follows from the definition of m that $1 \leqslant c\lambda_2^m/\lambda_1^n \leqslant 1/\lambda_2$, and hence each rectangle $R^*_{(i_1,\ldots,i_n)}$ projects onto a *disjoint* interval (on the horizontal axis) of length $c\lambda_2^m$. It thus contains at least one interval of length λ_1^n and can be covered by $1/\lambda_2$ intervals of length λ_1^n. This implies that

$$2^m \leqslant \#\{\lambda_1^n \text{ intervals needed to cover } \pi_2(F \cap F^*_m)\} \leqslant \frac{1}{\lambda_2} 2^m$$

i.e., $N(\lambda_1^n) \asymp 2^m$. Hence

$$\frac{\log N(\lambda_1^n)}{\log(1/\lambda_1^n)} \asymp \frac{\log(2^m)}{\log(1/\lambda_1^n)} \asymp \frac{\left[\dfrac{\log(1/\lambda_1^n)}{\log(1/\lambda_2)}\right] \log 2}{n \log(1/\lambda_1)} \xrightarrow{n \to \infty} \frac{\log 2}{\log(1/\lambda_2)}$$

The proposition follows from Lemma A1. ∎

The following result follows immediately from the calculation of $\dim_B F$:

Proposition 4. If one avoids the degenerate configurations in case 1, then for fixed λ_1, R_0, and R_1, the map $\lambda_2 \to \dim_B F(\lambda_2)$ is Lipschitz but not differentable.

Remark. This type of phenomenon, where a fundamental invariant changes in a Lipschitz but not smooth way, is quite rare in hyperbolic dynamical systems.

4. CALCULATION OF $\dim_H F$

In this section we turn to the problem of computing the Hausdorff dimension of the set F. Our formulas are based on some number-theoretic properties of the value λ_2 and should be compared with those estimates in ref. 7.

Let β be any real number between 0 and 1. For any $n \geqslant 1$ consider the set

$$J_n = \{0, 1\}^n$$

consisting of 2^n elements.

Define the map $p_n: J_n \to [0, 1/(1-\beta)]$ by

$$p_n((i_0,..., i_{n-1})) = \sum_{r=0}^{n-1} i_r \beta^r$$

Definition. The number β satisfies condition **GE** (after Garcia–Erdos) if there exists a constant $C > 0$ such that for all $x \in [0, +\infty)$ we have

$$A_\beta(n) = \text{Card}\{(i_0,..., i_{n-1}): p_n(i_0,..., i_{n-1}) \in [x, x+\beta^n)\} \leqslant C(2\beta)^n$$

Remark. We can also use the slightly weaker assumption that for all $\beta' > \beta$, choose a constant $C = C(\beta')$ with the above properties.

Lemma 1. If $0 < \beta < \frac{1}{2}$, then $A_\beta(n) = 1$. Hence condition **GE** is violated for all $0 < \beta < \frac{1}{2}$.

Proof. Observe that in this case the maps p_n are bijective onto their images. Furthermore, the points in the image are separated by a distance of at least β^n. The result follows easily. ■

Remarks. 1. There exist values $\frac{1}{2} < \beta < 1$ such that condition **GE** does not hold. Consider the case where $1/\beta$ is the golden mean, i.e.,

$1+1/\beta=1/\beta^2$. If we consider the 2^n strings $(w_1,..., w_n)\in J_{3n}$, where each $w_i=(1,0,0)$ or $(0,1,1)$, then it is clear that they have the same images under p_{3n} and thus $A_\beta(3n)\geqslant 2^n$. However, it is an easy numerical check that $2>(2\beta)^3=1.88854...$, which clearly contradicts the **GE** assumption.

2. In Appendix B we show that if β is the reciprical of a root of 2, then β satisfies condition **GE**. We also show that the set of **GE** numbers in the intervals $[\frac{1}{2}, 1]$ has positive Lebesgue measure.

Proposition 5. 1. If $0<\lambda_2<\frac{1}{2}$, then $\dim_B F=\dim_H F=\log 2/\log(1/\lambda_2)$.

2. If $\frac{1}{2}<\lambda_2<1$ satisfies condition **GE**, then

$$\dim_B F=\dim_H F=\frac{\log(2\lambda_2/\lambda_1)}{\log(1/\lambda_1)}$$

Proof. Let B_δ be a ball of radius δ. Choose $k\in\mathbb{N}$ such that $\delta<\lambda_2^k<10\delta$ (e.g., $k=[\log\lambda_2/\log\delta]+1$), and choose $m=[k\log(1/\lambda_2)/\log(1/\lambda_1)]$. Clearly $\lambda_1^m\asymp\lambda_2^k$. Using an idea in ref. 14, we consider the "asymptotic squares" S of dimensions $\lambda_1^m\times\lambda_2^k$ that prolongate each rectangle in F_k. There are two asymptotic squares associated to each rectangle in F_k, corresponding to "prolongating to the left" and "prolongating to the right."

It is convenient to break up the rest of the proof into four shorter lemmas.

Lemma 2. There exists a bound $C\geqslant 1$ [independent of δ and $k=k(\delta)$] such that $F\cap B_\delta$ can be covered with at most C asymptotic squares (Fig. 2).

Proof. This is a simple piece of geometry in the plane. ■

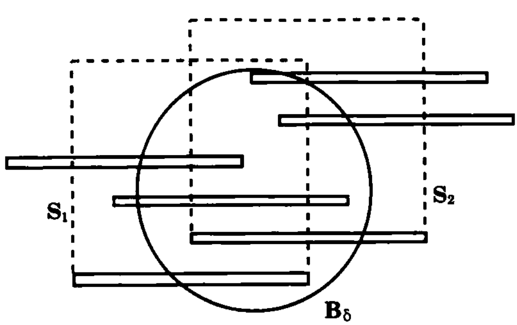

Fig. 2. $F\cap B_\delta$.

We now apply the mass distribution principle (see Appendix A). It follows from Lemma 2 that for any probability measure μ supported on the limit set F,

$$\mu(B_\delta) \leqslant \sum_{\text{bdd } \# \text{ of } S} \mu(S) \tag{4.1}$$

where the sum is over at most C asymptotic squares, and

$$\mu(S) \leqslant \sum_{R \in F_k \ni R \cap S \neq \varnothing} \mu(R) \tag{4.2}$$

where the sum is over those rectangles R in F_k that intersect the asymptotic square S.

Let $\mu = (1/2, 1/2)^{\mathbb{N}}$ be the equidistributed Bernoulli measure, i.e., each rectangle $R \in F_k$ has mass $\mu(R) = (1/2)^k$. Using (4.2), we have a bound on the μ-measure of S of the form

$$\mu(S) \leqslant N(k)(\tfrac{1}{2})^k, \qquad \text{where} \quad N(k) = \max \#\{R \cap S\}$$

Hence, using (4.1), we get a bound on the μ-measure of asymptotic squares

$$\mu(B_\delta) \leqslant C \max \mu(S) \leqslant CN(k)(\tfrac{1}{2})^k$$

Lemma 3. If $N(k)$ is uniformly bounded in k, then $\dim_{\mathrm{H}} F \geqslant \log 2/\log(1/\lambda_2)$.

Proof. Notice that $(1/2)^k = \lambda_2^{ks}$, where $s = \log 2/\log(1/\lambda_2)$. Hence

$$\mu(B_\delta) \leqslant CN(k)(\tfrac{1}{2})^k \leqslant CM\lambda_2^s \leqslant CM \cdot 10^s \delta^s$$

The lemma follows immediately from the mass distribution principle. ■

Remark. We actually proved a stronger result: the lower pointwise dimension[21] $\underline{\dim}_\mu^L(x) \geqslant \log 2/\log(1/\lambda_2)$ for all $x \in F$.

Lemma 4. If $N(k) \leqslant K(2\lambda_2)^{k-m}$, then

$$\dim_{\mathrm{H}} F \geqslant \frac{\log(2\lambda_2/\lambda_1)}{\log(1/\lambda_1)}$$

Proof. We estimate

$$\begin{aligned}\mu(B_\delta) &\leqslant CN(k)(\tfrac{1}{2})^k \leqslant C(2\lambda_2)^{k-m}(\tfrac{1}{2})^k = CK\lambda_2^k(2\lambda_2)^{-m} \\ &= CK\lambda_2^k(2\lambda_2)^{-k\log\lambda_2/\log\lambda_1} = CK\lambda_2^k\lambda_2^{-k\log(2\lambda_2)/\log\lambda_1}\end{aligned}$$

If $s=\log(2\lambda_2/\lambda_1)/\log(1/\lambda_1)$, then

$$10^s\delta^s \geqslant \lambda_2^{ks} = \lambda_2^k \lambda_2^{-k\log(2\lambda_2)/\log\lambda_1}$$

The lemma now follows immediately from the mass distributuion principle in Appendix A. ■

Remark. As in Lemma 3, we actually proved a stronger result: the lower pointwise dimension $\underline{\dim}_\mu^L(x) \geqslant \log(2\lambda_2/\lambda_1)/\log(1/\lambda_1)$ for all $x\in F$.

Consider a rectangle $R=R_{(i_1,\dots,i_k)}\in F_k$ and the corresponding asymptotic square S that prolongates R.

To complete the proof of Proposition 5, we now obtain the bounds for $N(k)$ in Lemmas 3 and 4.

Lemma 5. The other rectangles $R'\in F_k$ such that $R'\cap S\neq\emptyset$ have the codings of the form

$$(\underbrace{i_1,\dots,i_m}_{\text{fixed}}, j_{m+1},\dots,j_k)$$

Proof. For the rectangles R and R' to intersect in a common asymptotic square S, their separation can be at most λ_1^m. ■

We want to estimate the number of rectangles in the intersection. The left endpoint of $R_{(i_1,\dots,i_k)}$ is $\sum_{l=0}^k(\varepsilon_1+d\lambda_2^l)\,\lambda_2^l$, and the left endpoint of $R'=R_{(i_1,\dots,i_m,j_{m+1},\dots,j_k)}$ is $\sum_{l=0}^m(\varepsilon_1+di_l)\,\lambda_2^l+\sum_{l=m+1}^k(\varepsilon_1+dj_l)\,\lambda_2^l$. Hence for R and R' to lie in the same asymptotic square S, we require that

$$d\left|\sum_{l=m+1}^k (i_l-j_l)\,\lambda_2^l\right| < \lambda_2^k$$

Lemma 6. Given any sequence $p\in\prod_0^\infty\{0,1\}$, let

$$N^p(k)=\left\{(i_1,\dots,i_k)\in\{0,1\}^k : d\left|\sum_{l=M+1}^k (p_l-i_l)\,\lambda_2^l\right| < \lambda_2^k\right\}$$

1. If $0<\lambda_2<\frac{1}{2}$, then there exist $M, N>0$, such that $N^p(k)\leqslant M$ for all sequences p and $A_{\lambda_2}(k)\leqslant M$.

2. If $\frac{1}{2}<\lambda_2<1$, then λ_2 is a **GE** number if and only if for all sequences p, there exists $K>0$ such that $N^p(k)\leqslant K(2\lambda_2)^{k-m}$.

Proof. The proof of part 1 follows from the estimate on $A_\beta(n)$ in Lemma 1, with the choice $\lambda_2=1/\beta$. In particular, we can let $x_{M,k}=\sum_{l=M+1}^k p_l/\beta^l$. Then by Lemma 1 there are no other expansions within distance $1/\beta^n$. To complete the proof, we need only repeat this observation where we replace the choice of $x_{M,k}$ with translates by λ_2^k. This requires at most $[1/d]+1$ such translates, from which we deduce that $N^p(k) \leqslant ([1/d+1])\, A_\beta(k)$.

The proof of part 2 is very similar. ∎

Remarks. 1. We can give an easy alternative proof of Proposition 5, part 1. It follows from well-known properties of Hausdorff dimension[5] that $\dim_H(\pi_2(F)) \leqslant \dim_H(F) \leqslant \dim_B(F)$. Since $0 \leqslant \lambda_2 \leqslant \frac{1}{2}$, Corollary 1.1 and Proposition 3 imply that $\pi_2(F)$ is a uniform Cantor set constructed using 2^k disjoint intervals of length $d\lambda_2^{k+1}/(1-\lambda_2)$ at step k. One easily computes the Hausdorff dimension of $\pi_2(F) = \log 2/\log(1/\lambda_2)$. The formula now easily follows from Proposition 3.

2. A heuristic explanation of why number-theoretic properties of λ_2 determine whether $\dim_H F = \dim_B F$ is the following: If λ_2 is a **GE** number, then the rectangles $R \in F_k$ are horizontally *well dispersed*, and at most a fixed percentage can intersect an asymptotic square. However, if λ_2 is a PV number, then the rectangles $R \in F_k$ tend to "bunch up" at various places, and *a priori*, *most* of the rectangles may intersect an asymptotic square.

5. CONDITION GE AND RANDOM GEOMETRIC SERIES

There is an intimate connection between the property **GE** and a famous classical problem about random geometric series or infinitely convolved Bernoulli measures (ICBMs). Let $\{X_n\}_{n=1}^{\infty}$ denote a family of independent and identically distributed Bernoulli random variables such that $P(\varepsilon_k = 0) = P(\varepsilon_k = 1) = \frac{1}{2}$ for $k = 1, 2,\ldots$. For β a real number between 0 and 1, consider the random variable

$$S = \sum_{n=1}^{\infty} \varepsilon_n \beta^n$$

It can be shown that the compactly supported distribution μ_S of S is of pure type, i.e., either totally singular or absolutely continuous with respect to Lebesgue measure λ on $[0, \beta/(1-\beta)]$. An important open problem is to characterize all values of β (Erdos numbers) such that μ_S is absolutely continuous. We refer the reader to ref. 1 for the fascinating history of this still largely unsolved problem along with an interesting application to dynamical systems.

If β satisfies condition **GE**, then for $x \in \mathbb{R}$,

$$\mu_s^n[x, x+\beta^n] = \frac{A_\beta(n)}{2^n} \leqslant C\beta^n = C\lambda[x, x+\beta^n]$$

Hence, $\mu_s = \lim_{n \to \infty} \mu_s^n$ is absolutely continuous with respect to Lebesgue measure, with a uniformly bounded Radon–Nikodyn derivative.

Conversely, if the distribution μ_s is absolutely continuous with uniformly bounded density, then β satisfies **GE** (cf. Appendix C).

Among those explicitly known values such that the distribution S gives an absolutely continuous measure are reciprocals of roots of 2 (see Appendix B) and some more algebraic numbers by Garsia.[9] Furthermore, **GE** holds for almost all β sufficiently close to 1 (cf. Proposition C1 in Appendix C).

A related condition has appeared in the work of several authors who attempt to compute the dimension of graphs of Weierstrass-like functions.[2,3] Actually, the example in Section 2, where the Hausdorff dimension and box dimension differed, is an example where our sets are graphs of such functions. The authors show that if the projection of a certain natural measure on the graph is absolutely continuous, then the box dimension of the graph coincides with the Hausdorff dimension.

6. GENERALIZATIONS

Up to this point we have chosen to concentrate on our "model problem." However, it is apparent that the method is somewhat more general. To illustrate this, we shall mention a few of the possible generalizations with indications of their proofs.

Proposition 6. Replace the two similar boxes by two (or more boxes) with longest sides $\lambda_{2,1}, \lambda_{2,2} \leqslant \frac{1}{2}$. If one avoids degenerate configurations where the projection of the smaller rectangle onto the λ_2 direction is symmetrically contained in the projection of the larger rectangle, then the Hausdorff dimension of the limit set F coincides with the box dimension of F and is equal to the unique value δ such that $\sum_i \lambda_{2,i}^{\delta} = 1$.

The proofs are exactly the same, except that the measure used in the mass distribution principle is now the Bernoulli measure $(\exp(-\lambda_{2,1}\delta)/S, \exp(-\lambda_{2,2}\delta)/S)$, where $S = \exp(-\lambda_{2,1}\delta) + \exp(-\lambda_{2,2}\delta)$.

A further generalization of this is the following proposition:

Proposition 7. Assume that B_0, B_1 are two rectangles and assume that the sides are each less than $\frac{1}{2}$ in length. Assume that we have contractions (with these images) of the form $A_i(x, y) = (f_i(x), \lambda_1 y)$, $i = 1, 2$ where f_1, f_2 are the inverse branches associated to a simple Markov map f on $\pi_1 F$, where $|f_1'|, |f_2'| \leqslant \frac{1}{2}$. Then $\dim_B F = \dim_H F$ and they equal the value δ characterized by $P(-\delta \log |f'|) = 0$, where P denotes the thermodynamic pressure.[20]

Remark. In the constructions we have considered, each $(i_1,..., i_n) \in \{0, 1\}^n$ corresponds to a unique rectangle $R_{i_1,...,i_n}$, and all 2^n rectangles at step n are used in the construction of the limit set. One can generalize this construction by restricting the set of allowable rectangles at step n. For instance, one case choose a stochastic transition matrix A and allow only A-allowable rectangles at step n, i.e., $R_{i_1,...,i_n}$, where $A_{i_l, i_{l+1}} = 1$ for $l = 1,..., n-1$. It is easy to check that in this case the formula for $\dim_B F$ and $\dim_H F$ in Sections 3 and 4 are simply modified so that the $\log 2$ occurring in the numerator is replaced by $\log \Lambda$, where Λ is the maximal eigenvalue for A.

Generalizations to r Dimensions

In this section, we compute the box dimension of the limit set in the case of two r-dimensional boxes in the r-dimensional cube. Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r$ denote the r contraction coefficients. We first discuss the three-dimensional case:

Case I. Assume that $\lambda_1 \geqslant \lambda_2 \geqslant \frac{1}{2}$ and that $\lambda_3 \leqslant \lambda_2$ has no constraints. Let $\varepsilon = \lambda_3^n$. We want to cover the limit set by cubes with sides ε.

If we go down to the nth level F_n of rectangles, we first observe that the assumption on λ_1, λ_2 implies that the projections of F onto the corresponding axes actually contain intervals. Thus, we can estimate the number of cubes required to cover *each* of the rectangles by

$$N = \frac{\lambda_1^n}{\lambda_3^n} \frac{\lambda_2^n}{\lambda_3^n}$$

Thus, the box dimension is

$$\dim_B F = \frac{\log(\lambda_1 \lambda_2 / \lambda_3^2)}{\log \lambda_3}$$

Case II. Assume $\lambda_1 \geqslant \frac{1}{2} \geqslant \lambda_2$. The difference now is that while the projection into the first axis still contains an interval, the projection into the second axis is a Cantor set. If we let $\varepsilon = \lambda_3^n$, then again the number of boxes of this size needed to cover the set F is estimated by

$$2^n \frac{\lambda_1^n}{\lambda_3^n} 2^M$$

where $M = M(n)$ is the value such that $\lambda_2^{n+M} = \lambda_3^n$. The appearance of M is because the projection of each rectangle in F_n into the second axis corresponds to an interval of size λ_2^n, and the projection of the Cantor set can be efficiently covered by 2^M intervals of size λ_3^n, where $M = n \log(\lambda_3/\lambda_2)/\log \lambda_2 = nM'$ satisfies the above condition.

The box dimension is then given by

$$\dim_B F = \frac{\log 2}{\log \lambda_3} + \frac{\log(\lambda_1/\lambda_3)}{\log \lambda_3} + \frac{\log 2}{\log \lambda_3}$$

Case III. Assume $\frac{1}{2} \geqslant \lambda_1 \geqslant \lambda_2$. Following the previous reasoning, we can expect to efficiently cover the limit set F by $N = 2^n 2^M 2^Q$ cubes with sides λ_3^n, where M is as in case II, and Q is a similar value, except we replace λ_2 by λ_1, i.e., $Q = n \log(\lambda_3/\lambda_2)/\log \lambda_2 = Q'n$. The formula for the box dimension then becomes

$$\dim_B F = \frac{\log 2}{\log \lambda_3} + M' \frac{\log 2}{\log \lambda_3} + Q' \frac{\log 2}{\log \lambda_3}$$

General Case. Let us assume that we now have two boxes in an r-dimensional cube, with contraction coefficients $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_s \geqslant \frac{1}{2} \geqslant \lambda_{s+1} \geqslant \cdots \geqslant \lambda_r$.

Define $m_i = \log(\lambda_r/\lambda_i)/\log \lambda_i$. Then the box dimension of the limit set F is equal to

$$\dim_B F = \frac{\log(\lambda_1/\lambda_r)}{\log \lambda_r} + \cdots + \frac{\log(\lambda_s/\lambda_r)}{\log \lambda_r} + \frac{\log 2}{\log \lambda_r} (m_{s+1} \cdots m_r + 1)$$

The Hausdorff dimension can be computed similarly provided the projections onto the slower axes are disjoint or satisfy special assumptions related to those before.

7. THE HAUSDORFF DIMENSION OF LINEAR HORSESHOES

The Cantor sets that arise in the model problem are closely related to an important class of diffeomorphisms. Specifically, it is easy to construct a Smale horseshoe diffeomorphism (with a two-dimensional stable manifold and a one-dimensional unstable manifold) for which the associated basic set Λ is a product of the limit set F constructed in the model problem and a uniform Cantor set E_p in the line. As usual, Smale horseshoes are constructed by specifying a box R and its image under the diffeomorphism. To arrange that the basic set is of the form described above, we require that the diffeomorphism f should be affine on $R \cap f^{-1}R$. This is illustrated in Fig. 3.

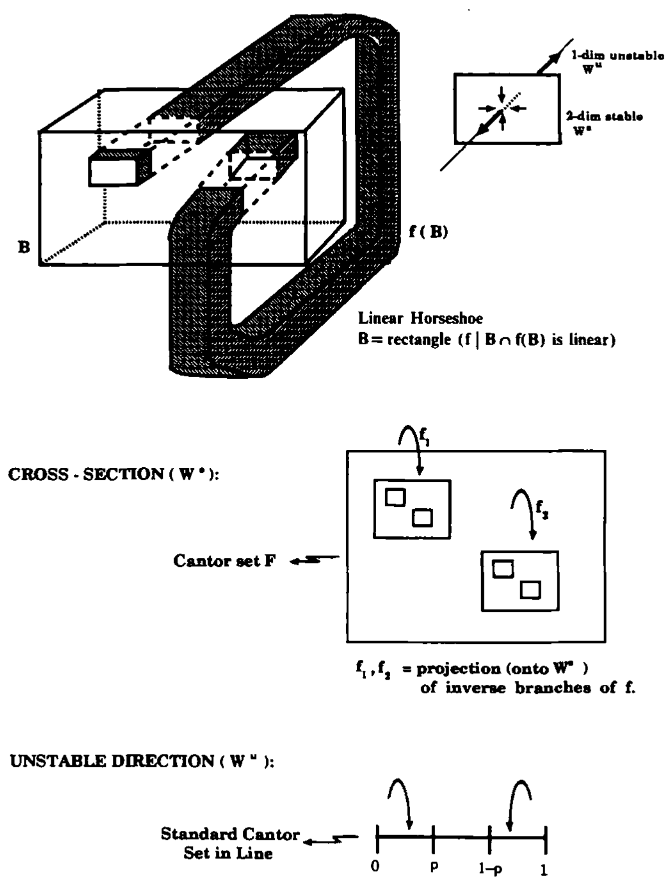

Fig. 3. Linear horsehoe.

Since $\Lambda = F \times E_p$, where E_p is a uniform Cantor set, it follows[10] that $\dim_H \Lambda = \dim_H F + \dim_H E_p$ and $\dim_B \Lambda = \dim_B F + \dim_B E_p$.

We wish to construct a horseshoe based on the Przytycki–Urbanski (PU) example mentioned in Section 2, where the dimensions of the limit set do not coincide. The PU example consists of two rectangles of width the reciprocal of the golden mean (which is easily seen to be a PV number) and height $\frac{1}{2}$, that are flush against the top left and bottom right corners of the unit square, respectively. Unfortunately, the rectangles in the PU example are not disjoint, and hence one cannot effect the horseshoe construction.

We wish to slightly perturb the PU example by shrinking the heights of the rectangles to $\lambda_1 < 1/2$, keeping the widths the recipricol of the golden mean, and keeping the new rectangles flush against the top left and bottom right corners of the unit square (respectively). See Fig. 4.

We wish to show that the dimensions of the limit set do not coincide for these perturbed examples. It immediately follows from our explicit formula for the box dimension (Proposition 2) that the box dimension of the limit sets of our modified PU examples changes smoothly as a function of λ_1. It is not the case that there is strict containment of the two limit sets.

The covering argument in ref. 16 showing that the Hausdorff dimension of the PU examples is strictly less than the box dimension applies without change to show that the Hausdorff dimension of the *modified* PU examples is strictly less than the box dimension. The argument is independent of λ_1. It follows that there exists an open interval around $\lambda_1 = \frac{1}{2}$ such that for λ_1 in this interval, the dimensions of the limit sets for the modified PU examples do not coincide. *This furnishes a one-parameter family of horsehoes whose dimensions do not coincide.*

Using obvious modifications of these constructions, we can arrange similar realizations, as basic sets, of limit sets generated by any number of affine maps, *provided that the image rectangles are disjoint.* We consider below two different types of cases where the dimensions can be explicitly computed, with interesting conclusions.

Example 1. Horseshoes with different dimensions. Choose a finite number n_1, say, of disjoint subintervals of the same length λ_1 in the interval $[0, 1]$. Next choose a finite number n_2 of disjoint subintervals in $[0, 1]$ with length λ_1 strictly smaller than λ_2. Consider the rectangles in the unit square corresponding to the products of these two sets of intervals.

Choose from this family a subfamily of k rectangles. In particular, this yields a finite number of disjoint rectangles such that (1) each rectangle has height λ_1 and width λ_2, (2) disjoint rectangles either have the same projection or disjoint projections onto the horizontal direction, and (3) disjoint rectangles either have the same projection or disjoint projections onto the

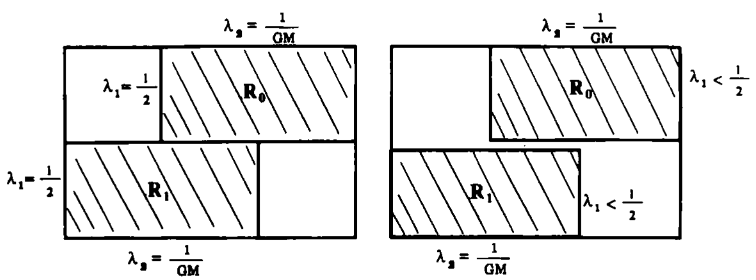

Fig. 4. Left: PU example; right: modified PU Example.

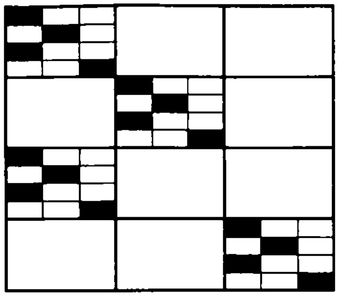

Fig. 5. Sierpinski carpet.

vertical direction. See Fig. 5. A Sierpinski-like limit set F is generated by the k affine contractions associated with this subfamily of rectangles. This construction is modeled by a full shift on k symbols.

This is a slight generalization of the familiar Bedford–McMullen problem,[2, 14] inasmuch as the vertices of the rectangles need not have rational coordinates. The Hausdorff and box dimensions of the limit set F of these (and more complicated) systems have been studied by Lalley and Gatzouras[13] and Kenyon and Peres.[10] The Hausdorff and box dimensions of these systems are known to be

$$\dim_{\mathrm{H}} F = \frac{1}{\log \lambda_2} \log \left[\sum_{i=1}^{n_2} (\log k_i)^{\log \lambda_1 / \log \lambda_2} \right]$$

where n_i is the number of rectangles from the subfamily left in the ith row, and

$$\dim_B F = -\frac{1}{n_2} \left(\frac{\log n_1}{\log \lambda_2} + \frac{\log(k/n_1)}{\log \lambda_1} \right)$$

We want to make our choices of $\lambda_1, \lambda_2 > 0$ and configuration of rectangles such that $\dim_{\mathrm{H}} F \neq \dim_B F$. Consider the rectangles $R_0 = [0, 1/2] \times [0, \alpha]$, $R_1 = [0, 1/2] \times [1-\alpha, 1]$, and $R_2 = [1/2, 1] \times [(1-\alpha)/2, (1+\alpha)/2]$, where $0 < \alpha < 1/3$. Clearly, for typical values of α we will have that $\dim_{\mathrm{H}} F \neq \dim_B F$.

If we consider an associated horseshoe Λ associated with the family of affine maps generating F, then we see that

$$\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{H}} F + \dim_{\mathrm{H}} E_p \neq \dim_B F + \dim_B E_p = \dim_B \Lambda$$

i.e., the associated horseshoe limit set Λ has Hausdorff dimension different from the box dimension.

Example 2. Dependence of dimension on configuration. Choose n_1 disjoint subintervals of the same length λ_1 in the interval $[0, 1]$. Next choose n_2 disjoint subintervals in $[0, 1]$ with length λ_1 strictly smaller than λ_2. Consider the rectangles in the unit square corresponding to the products of these two sets of intervals. We associate to these rectangles linear contractions and denote the corresponding limit set by Λ_1. It is easy to see that the Hausdorff dimension and box dimension of the *projection* of Λ_1 onto the horizontal axis coincide and equal $\log n_2/\log(1/\lambda_2)$. Similarly, we see that the Hausdorff dimension and box dimension of the *projection* of Λ_1 onto the vertical axis coincide and equal $\log n_1/\log(1/\lambda_1)$. It is then easy to see that since the space Λ_1 is a product of these projections, then

$$\dim_H \Lambda_1 = \dim_B \Lambda_1 = \frac{\log n_1}{\log(1/\lambda_1)} + \frac{\log n_2}{\log(1/\lambda_2)}$$

Assume that the product $N = n_1 \cdot n_2$ has a different factorization $N = m_1 \cdot m_2$. We can repeat the above construction of an array of boxes using m_1 disjoint subintervals of the same length λ_1 in the interval $[0, 1]$ and m_2 disjoint subintervals in $[0, 1]$ of equal length λ_2. We shall denote by Λ_2 the corresponding limit set, and by a similar reasoning we see that

$$\dim_H \Lambda_2 = \dim_B \Lambda_2 = \frac{\log m_1}{\log(1/\lambda_1)} + \frac{\log m_2}{\log(1/\lambda_2)}$$

If we take the specific choices $n_1 = 6$, $n_2 = 2$ and $m_1 = 4$, $m_2 = 3$, then for typical values of λ_1 and λ_2 we have (see Fig. 6)

$$\dim_H \Lambda_1 = \dim_B \Lambda_1 \neq \dim_H \Lambda_2 = \dim_B \Lambda_2$$

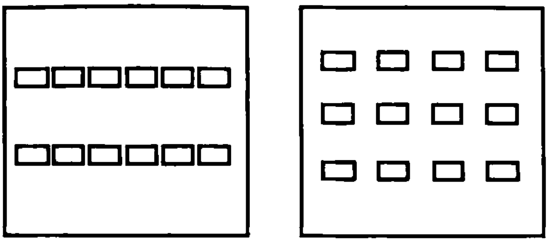

Fig. 6. Different configurations of 12 rectangles.

APPENDIX A. FACTS ABOUT DIMENSION THEORY

Definition. Let $U \subset \mathbb{R}^n$. The diameter of U is defined as $|U| = \sup\{|x-y|: x, y \in U\}$. If $\{U_i\}$ is a countable collection of sets of diameter at most δ that cover Z, i.e., $Z \subset \bigcup_i U_i$ with $0 < |U_i| \leqslant \delta$ for each i, we say that $\{U_i\}$ is a δ-cover of Z.

Suppose that $Z \subset \mathbb{R}^n$ and $s \geqslant 0$. For any $s > 0$, define

$$m_{\mathrm{H}}(s, Z) = \lim_{\delta \to 0} \inf_{\{U_i\}} \left\{ \sum_i |U_i|^s \colon \{U_i\} \text{ is a } \delta\text{-cover of } Z \right\}$$

We call $m_{\mathrm{H}}(s, Z)$ the *s-dimensional Hausdorff measure of* Z. There exists a unique critical value of s at which $m_{\mathrm{H}}(s, Z)$ jumps from ∞ to 0. This critical value is called the *Hausdorff dimension* of Z and is written $\dim_{\mathrm{H}}(Z)$. If $s = \dim_{\mathrm{H}}(Z)$, then $m_{\mathrm{H}}(s, Z)$ may be 0, ∞, or finite. Hence $\dim_{\mathrm{H}}(Z) = \sup\{s \colon m_{\mathrm{H}}(s, Z) = \infty\} = \inf\{s \colon m_{\mathrm{H}}(s, Z) = 0\}$.

Definition. Let $N_\delta(Z)$ denote the minimum of sets of diameter precisely δ needed to cover the set Z. We define the upper and lower box dimension measures of Z by

$$\underline{\dim}_{\mathrm{B}} Z = \liminf_{\delta \to 0} \frac{N_\delta(Z)}{\log(1/\delta)}$$

$$\overline{\dim}_{\mathrm{B}} Z = \limsup_{\delta \to 0} \frac{N_\delta(Z)}{\log(1/\delta)}$$

If $\underline{\dim}_{\mathrm{B}} Z = \overline{\dim}_{\mathrm{B}} Z$, denote the common value by $\dim_{\mathrm{B}} Z$.

Remark. It is easy to see that $\dim_{\mathrm{H}}(Z) \leqslant \underline{\dim}_{\mathrm{B}}(Z) \leqslant \overline{\dim}_{\mathrm{B}}(Z)$. The usual method of obtaining an upper bound for $\dim_{\mathrm{H}}(Z)$ is to obtain an upper bound for $\underline{\dim}_{\mathrm{B}}(Z)$.

The following proposition is extremely useful for obtaining a lower bound for the Hausdorff dimension of a set:

Mass Distribution Principle (Frosteman).[5] Let μ be a measure supported on Z and suppose that for some s there are numbers $c > 0$ and $\delta > 0$ such that $\mu(U) \leqslant c\,|U|^s$ for all sets U with $|U| \leqslant \delta$. Then $m_{\mathrm{H}}(s, Z) \geqslant \mu(Z)/c$ and $s \leqslant \dim_{\mathrm{H}}(Z)$.

The following simple lemma shows that in computing the box dimension of a set we need only consider the minimum number $N(\delta)$ of covering boxes where δ runs through a geometric sequence converging to zero.

Lemma A1.[5] Fix $0<c<1$. Then

$$\dim_{\mathrm{B}} Z=\lim_{\delta\to 0}\frac{\log N(\delta)}{\log(1/\delta)}=\lim_{k\to\infty}\frac{\log N(c^k)}{\log(c^k)}$$

APPENDIX B. CONDITION GE FOR RECIPRICALS OF ROOTS OF 2

We give some easy examples of numbers $1<\beta\leqslant 2$ that show the **GE** condition can sometimes be checked without resorting to the more complicated analysis on the projection of the measure.

1. In the special case where $\beta=2$, the **GE** condition is easy to check, since this is a question about diadic expansions.
2. If $\beta=2^{1/2}$ and n is even, then we can write

$$\begin{aligned}\sum_{r=0}^{n-1}\frac{i_r}{\beta^r}&=\sum_{r=0}^{n/2-1}\frac{i_{2r}}{2^r}+\frac{1}{\beta}\left(\sum_{r=0}^{n/2-1}\frac{i_{2r+1}}{2^r}\right)\\&=A+\frac{1}{\beta}B\end{aligned}$$

For this expression to lie in an interval $[x, x+1/\beta^n)$, we have at most $2^{n/2}$ choices for the value A and a bounded (one) number of choices for B. This suggests that we have a bound for the above expression of the general form

$$C2^{n/2}=C\left(\frac{2}{\beta}\right)^n$$

as required for the **GE** condition.

A similar argument works for any root of 2.

APPENDIX C. CONDITION GE FOR SETS OF POSITIVE MEASURE

In this appendix, we explain the relationship between the condition **KS** in ref. 11, p. 198, our condition **GE**, and the work of Erdos.[4]

Definition. The number β satisfies condition **KS** if there exists a constant $C>0$ such that the number $N_\beta(n)$ of solutions to $|p_n(i_0,..., i_{n-1})-p_n(i_0,..., j_{n-1})|\leqslant 1/\beta^n$ satisfies $N_\beta(n)\leqslant C(4\beta)^n$ (ref. 11, p. 198).

Remark. It is easy to see that the condition **GE** implies condition **KS**.

The usefulness of this assumption is shown by the following results.

Proposition C1. 1. A number β satisfies condition **KS** if and only if the Fourier transform $\gamma(t) = \int e^{2\pi i x t}\, d\mu_s(x)$ of the measure μ_s is in $L^2(\mathbb{R})$, where μ_S is the distribution of the random variable $S = \sum_{n=1}^{\infty} \varepsilon_n \beta^n$ (see Section 5) (ref. 11, pp. 197–198).

2. For each positive integer $m \geqslant 1$ there exists $\delta > 0$ such that for almost all $\beta \in [1-\delta, 1)$ (in the sense of Lebesgue) we have $\gamma(t) = O(1/|t|^m)$ as $|t| \to +\infty$. When $m = 1$ we have that $\gamma \in L^2(\mathbb{R})$ (ref. 4, p. 186).

We can conclude that for almost all values of β sufficiently close to 1 the condition **KS** holds. We would like to reach the same conclusion for the stronger condition **GE**. We begin with the following lemma.

Lemma C1. There exists $\delta > 0$ such that for almost all $\beta \in [1-\delta, 1)$ the distribution μ_s is absolutely continuous with a uniformly bounded continuous density.

Proof. By part 2 of Proposition C1, we can choose δ such that the Fourier transform $\gamma(t)$ is in $L^1(\mathbb{R})$ for almost all $\beta \in [1-\delta, 1)$. Fix such a value of β.

It immediately follows from the Fourier inversion formula and the Riemann–Lebesgue lemma that if the Fourier transform $\gamma(t)$ of the distribution μ_s is in $L^1(\mathbb{R})$, then the distribution μ_s is absolutely continuous with a uniformly bounded continuous density h defined by $h(x) = \int e^{-2\pi i x t} \gamma(t)\, dt$. ■

Proposition C2. There exists $\delta > 0$ such that for almost all $\beta \in [1-\delta, 1)$ the condition **GE** holds.

Proof. Choose δ as in the previous lemma. It remains to show that the density h being uniformly bounded implies that the condition **GE** holds. William Parry showed us a derivation of this using an analysis of fat baker's transformations in ref. 1. We shall present two additional proofs. The first proof is based on the study of transfer operators, which could be a useful new tool in the study of these number-theoretic problems. The second proof, which is considerably more elementary, was supplied to us by Yuval Peres.

We first present a proof based on the transfer operator

$$Lh(x) = \tfrac{1}{2}[h(x\beta) + h((x+1)\beta)]$$

defined on functions of bounded variation on $\mathbb{R}$, as described in ref. 11, p. 200.

The first observation is that for any such h we can expand

$$L^n h(x) = \sum_{(i_0, \dots, i_{n-1})} \frac{1}{2^n} h\left(\sum_{i_0, \dots, i_{n-1}} i_r \beta^r + x\beta^n\right)$$

and then observe the identity $L^n \chi_{[a, a+\beta^n]}(0) = N_\beta(n)/2^n$.

It is clear that the asymptotics of the function $N_\beta(n)$ are determined by the spectral properties of the operator L on the space of functions of bounded variation.

1. The operator has a maximal eigenvalue equal to unity, with an eigenprojection corresponding to the measure μ_s, i.e., $L^n k \to \int k \, d\mu_s$.

2. The remainder of the spectrum is contained within the disc about zero of radius β. By ref. 17 there can be only *isolated* eigenvalues α of modulus greater than β. However, since the operator also preserves the space of C^1 functions, the associated eigenfunctions $Lk = \alpha k$ must be C^1. By differentiating, we see that $(Lk)' = \beta L(k') = \alpha k'$, and thus α/β is also an eigenvalue, except where $\alpha = 1$ and k is the constant function. However, since the spectral radius of L is unity, we require that $|\alpha|/\beta \leqslant 1$, which completes the proof.

Therefore, we can write $L^n k = \int k \, d\mu_s + U^{(n)} k$, where $U^{(n)}$ is a linear operator with $\limsup_{n \to +\infty} \|U^{(n)}\|^{1/n} \leqslant \beta$.

We can write

$$\mu_s([a, a+\beta^n]) + U^{(n)} \chi_{[a, a+\beta^n]}(0) = L^n \chi_{[a, a+\beta^n]}(0) = \frac{N_\beta(n)}{2^n}$$

If we observe that the norms of $\chi_{[a, a+\beta^n]}$ are uniformly bounded (in the space of functions of bounded variation), we see that $U^{(n)} \chi_{[a, a+\beta^n]}(0) = O((\beta')^n)$ for any $\beta' > \beta$. It only remains to use the fact that the density h is uniformly bounded to see that $\limsup_{n \to +\infty} \{\mu[a, a+\beta^n])/\beta^n\}$ is finite. This completes the first proof. ■

The second proof is a proof by contradiction. If β is not **GE**, then there exist increasing sequences of integers n_i and C_i and intervals J_{n_i} such that $\lambda(J_{n_i}) = \beta^{n_i}$ and

$$\text{Card}\left\{(i_0, \dots, i_{n_i - 1}) \colon \sum_{r=0}^{n_i - 1} i_r \beta^r \in J_{n_i}\right\} \geqslant C_i (2\beta)^{n_i}$$

Since $\sum_{k=n}^{\infty} x^k = x^n/(1-x)$, we may symmetrically enlarge the intervals J_{n_i} to intervals $I_{n_i} \supset J_{n_i}$, $\lambda(I_{n_i}) = \beta^{n_i} + \beta^{n_i}/(1-\beta) = l\beta^{n_i}$, such that if $\sum_{r=0}^{n_i-1} i_r \beta^r \in J_{n_i}$, then $\sum_{r=0}^{\infty} i_r \beta^r \in I_{n_i}$. We obtain that

$$\operatorname{Card}\left\{(i_0,\dots, i_{n_i},\dots, i_{n_i+p}) \colon \sum_{r=0}^{n_i+p} i_r \beta^r \in I_{n_i}\right\} \geqslant 2^p C_i (2\beta)^{n_i}$$

This implies that

$$\frac{\mu_S(I_{n_i})}{\lambda(I_{n_i})} = \frac{\lim_{p\to\infty} [(\operatorname{Card}\{(i_0,\dots, i_{n_i},\dots, i_{n_i+p}) \colon \sum_{r=0}^{n_i+p} i_r \beta^r \in I_{n_i}\})/2^{n_i+p}]}{\lambda(I_{n_i})}$$

$$\geqslant \frac{C_i \beta^{n_i}}{l\beta^{n_i}} = \frac{C_i}{l}$$

which is unbounded as $i \to \infty$. It follows that the Radon–Nikodyn derivative $d\mu_S/d\lambda$ is not uniformly bounded. ■

We conjecture that (Lebesgue) almost all numbers $\frac{1}{2} < \beta < 1$ satisfy condition **GE**. By Proposition 5, part 2, this conjecture would imply that for almost all $\frac{1}{2} < \lambda_2 < 1$, the Hausdorff dimension of the limit set of the model similarity process coincides with the box dimension and is given by the formula in Proposition 5, part 1. Moreover, by Proposition 5, part 1, this conjecture would imply that for almost all $0 < \lambda_2 < 1$, the Hausdorff dimension coincides with the box dimension.

ACKNOWLEDGMENTS

M. P. would like to thank INIC (Portugal) and the Royal Society (England) for their support. H. W. was partially supported by a National Science Foundation Postdoctoral Research Fellowship and JNICT grant PMCT/C/CEN 35/90. The authors also thank William Parry for helpful discussions on fat baker's transformations and Yuval Peres for carefully reading and commenting on a draft of this paper and supplying many references to the bibliography. H. W. would like to thank Yakov Pesin for many stimulating discussions on dimension theory.

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